MATH 140A Review: Helpful Algebraic techniques

1. Rewrite $\sqrt{2}+\sqrt{3}$ as a fraction with no radicals in the numerator.

Solution: We have that

$$(\sqrt{2}+\sqrt{3})\frac{(\sqrt{2}-\sqrt{3})}{(\sqrt{2}-\sqrt{3})} = \frac{2-3}{(\sqrt{2}-\sqrt{3})} = \frac{-1}{(\sqrt{2}-\sqrt{3})}.$$

2. Rewrite $\sqrt{n+2}-\sqrt{n-1}$ as a fraction with no radicals in the numerator and then compute the limit as $n \to \infty$.

Solution: Since we notice that $\sqrt{n+2}-\sqrt{n-1}$ is of the form $x-y$, we will multiply by

$$\frac{\sqrt{n+2}+\sqrt{n-1}}{\sqrt{n+2}+\sqrt{n-1}} = 1,$$

to get

$$\begin{aligned}(\sqrt{n+2}-\sqrt{n-1})\cdot 1 &= (\sqrt{n+2}-\sqrt{n-1})\cdot\frac{\sqrt{n+2}+\sqrt{n-1}}{\sqrt{n+2}+\sqrt{n-1}}\\ &= \frac{(\sqrt{n+2}-\sqrt{n-1})\cdot(\sqrt{n+2}+\sqrt{n-1})}{\sqrt{n+2}+\sqrt{n-1}}\\ &= \frac{(n+2)-(n-1)}{\sqrt{n+2}+\sqrt{n-1}}.\end{aligned}$$

Notice that the trick gets rid of the radicals at the top. We then have

$$(\sqrt{n+2}-\sqrt{n-1}) = \frac{3}{\sqrt{n+2}+\sqrt{n-1}}.$$

Hence, $\sqrt{n+2}-\sqrt{n-1} \to 0$ as $n \to \infty$.

3. What is the limit of $a_n = n^{2/n}$ as $n \to \infty$?

Solution: We have that

$$a_n = n^{2/n} = e^{\ln\left(n^{2/n}\right)} = e^{2/n\ln(n)}.$$

By using L'hopital's rule, we get

$$\lim_{n\to\infty}\frac{2\ln n}{n} = 2\lim_{n\to\infty}\frac{1}{n} = 0.$$

Since the function e^x is continuous, then we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{2/n\ln(n)} = e^{\lim_{n\to\infty} 2/n\ln(n)} = e^0 = 1.$$